TALENT DEVELOPMENT CENTRE
INDIAN INSTITUTE OF SCIENCE, KUDAPURA
Challakere, Chitradurga District, Karnataka-577536

Problems on continuity and derivatives

1. Find the set of points of continuity of the function $f(x) = [x]$, where $[x]$ denotes the largest integer not exceeding x.

2. Give an example of a function on $\mathbb{R}$ which is not continuous at any point.

3. Find which of the following functions is continuous at the indicated point:

$$f(x) = \begin{cases} \frac{2x^2-3x-2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2, \end{cases} \text{ at } x = 2; \; f(x) = \begin{cases} |x| \cos \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases} \text{ at } x = 0.$$

4. Find the value of k which makes the function

$$f(x) = \begin{cases} \frac{2^{x+2}-16}{4^x-16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2, \end{cases} \quad \text{at } x = 2.$$

5. Let $f(x) = (x+1)/(x-3)$. Find the points of discontinuity of $f(f(x))$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = f(x^2)$ for all real x. Prove that f is a constant function.

7. Let $f : [a, b] \to [a, b]$ be a continuous function. Prove that there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

8. A person climbs from valley to a temple on a mountain. The next day, he climbs down along the same path during the same time interval. Prove that there is a point on his path that the person reached at precisely the same moment on both the days.

9. Prove that the equation $(1-x)\cos x = \sin x$ has at least one solution in the interval $(0, 1)$.

10. Let $n \in \mathbb{N}$ and let $f : [0, n] \to \mathbb{R}$ be a continuous function. Prove that there exists $x_1, x_2 \in [0, n]$ such that $x_2 - x_1 = 1$ and $f(x_1) = f(x_2)$.

11. Prove that the perimeter of a circle of radius r is $2\pi r$ and its area is πr^2.

12. In a cross-country race, a runner runs 6 km in exactly 30 minutes. Prove that some where along the course, the runner ran 1 km in exactly 5 minutes.

13. Let $f : (a, b) \to \mathbb{R}$ be a continuous function. Prove that given points $x_1, x_2, \ldots, x_n$ in (a, b), there exists a point x_0 in (a, b) such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

14. Let f be a real continuous function on $[0, 2]$ such that $f(0) = f(2)$. Prove that there exists x_1, x_2 in $[0, 2]$ such that

$$x_2 - x_1 = 1, \text{ and } f(x_2) = f(x - 1).$$

15. Suppose f and g have intermediate value property on $[a, b]$. Must $f + g$ possess intermediate value property on $[a, b]$?

16. Check whether the derivative exists and if so find it:
(a) $f(x) = x^2|x|$; (b) $f(x) = \sqrt{|x|}$, $x \in \mathbb{R}$; (c) $f(x) = [x]\sin^2(\pi x)$;
(d) $f(x) = \log_x 2$, $x > 0, x \neq 1$; (e) $f(x) = \begin{cases} x^2 e^{-x^2} & \text{if } |x| \leq 1, \\ \frac{1}{e} & \text{if } |x| > 1. \end{cases}$

17. Determine the constants a, b, c, d so that f is differentiable on $\mathbb{R}$;
(i) $f(x) = \begin{cases} 4x & \text{if } |x| \leq 0, \\ ax^2 + bx + c & \text{if } 0 < |x| < 1 \\ 3 - 2x & \text{if } |x| \geq 1; \end{cases}$ (ii) $f(x) = \begin{cases} ax + b & \text{if } |x| \leq 0, \\ cx^2 + dx & \text{if } 0 < |x| \leq 1 \\ 1 - \frac{1}{x} & \text{if } |x| > 1; \end{cases}$

18. Prove that $x^{13} + 7x^3 - 5 = 0$ has exactly one real root.