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Problems on derivatives

1. Check whether the derivative exists and if so find it:
   (a) $f(x) = x^2|x|$; (b) $f(x) = \sqrt{|x|}$, $x \in \mathbb{R}$; (c) $f(x) = [x]\sin^2(\pi x)$;
   (d) $f(x) = \log_x 2$, $x > 0, x \neq 1$; (e) $f(x) = \begin{cases} x^2 e^{-x^2} & \text{if } |x| \leq 1, \\ \frac{1}{e} & \text{if } |x| > 1. \end{cases}$

2. Determine the constants a, b, c, d so that f is differentiable on $\mathbb{R}$;
   (i) $f(x) = \begin{cases} 4x & \text{if } |x| \leq 0, \\ ax^2 + bx + c & \text{if } 0 < |x| < 1 \\ 3 - 2x & \text{if } |x| \geq 1; \end{cases}$ (ii) $f(x) = \begin{cases} ax + b & \text{if } |x| \leq 0, \\ cx^2 + dx & \text{if } 0 < |x| \leq 1 \\ 1 - \frac{1}{x} & \text{if } |x| > 1; \end{cases}$
   (iii) $f(x) = \begin{cases} \cos t & \text{if } t \geq 0, \\ a + bt + t^2 & \text{if } t < 0. \end{cases}$

3. Find a and b such that $\lim_{x \to 0} \dfrac{x(a\cos x + 1) - b\sin x}{x^3}$ exists and equal to 1.

4. Find a ploynomial $P(x)$ of the least degree such that $P(0) = 1$, $P(1) = 2$, $P'(0) = 3$, $P'(1) = 4$.

5. Using derivatives, prove that $x^3 + 3qx + r = 0$ has a double root if $r^2 + 4q^3 = 0$.

6. Prove that $x^{13} + 7x^3 - 5 = 0$ has exactly one real root.

7. Check whether the following functions are differentiable at 0:
   (i) $f(x) = \begin{cases} x^2 \sin(1/x), & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$ (ii) $f(x) = \begin{cases} x^2, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$
   (iii) $f(x) = \begin{cases} \cos x, & \text{for } x \geq 0, \\ 1 + x, & \text{for } x < 0. \end{cases}$ (iv) $f(x) = \begin{cases} e^x, & \text{for } x \geq 0, \\ 1 + x, & \text{for } x < 0. \end{cases}$

8. Verify mean value theorem for $f(x) = x(x-1)(x-2)$, $0 \leq x \leq \frac{1}{2}$.

9. Show that the equation

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = 0$$

has no real root for n even and has exactly one real root for n odd.

10. Show that if $f(x)$ is differentiable on an interval (a, b), then it has intermediate value property on every interval $[c, d]$ contained in (a, b).

11. Find the following limits:
(i) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\ln(1 + x)}$; (ii) $\lim_{x \to a} \frac{x - a}{e^x - e^a}$;
(iii) $\lim_{x \to 4} \left(\frac{1}{\log(x - 3)} - \frac{1}{x - 4} \right)$; (iv) $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$.

12. Find all points at which the function $f : (0, 2) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational;} \\ 2x - 1, & \text{if } x \text{ is irrational,} \end{cases}$$

is differentiable.

13. Show that among all rectangles of a given perimeter square encloses maximum area.

14. Find the points of local maxima, local minima and the points of inflection of the curve $y = x^5 - 5x^4 + 5x^3 - 1$. Find also the corresponding local maximum and local minimum values.

15. Prove that for all $x \neq 0$, $e^x > 1 + x$.

16. Let a_1, a_2, a_3 be real numbers and consider

$$f(x) = (x - a_1)(x - a_2) + (x - a_2)(x - a_3) + (x - a_3)(x - a_1).$$

Prove that $f(x) \geq 0$ for all $x \in \mathbb{R}$ if and only if $a_1 = a_2 = a_3$.

17. Find all real solutions of the equation

$$6^x + 1 = 8^x - (27)^{x-1}.$$