TALENT DEVELOPMENT CENTRE
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Problems on limits

1. Define a sequence $\langle a_n \rangle$ by

$$a_n = -2\sqrt{n} + \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \right).$$

   Prove that the sequence $\langle a_n \rangle$ converges.

2. Define a sequence $\langle a_n \rangle$ by $a_1 = 0$ and $a_{n+1} = \sqrt{6 + a_n}$ for $n \geq 1$. Prove that $\langle a_n \rangle$ converges.

3. Suppose $\langle a_n \rangle$ is a sequence which satisfies

$$0 < a_n < 1, \quad a_n(1 - a_{n+1}) > \frac{1}{4} \text{ for all } n \in \mathbb{N}.$$

   Prove that the sequence $\langle a_n \rangle$ converges and find its limit.

4. Find the following limits:

   (i) $\lim_{n\to\infty} \sqrt[n]{1^2 + 2^2 + \cdots + n^2}$; (ii) $\lim_{n\to\infty} \dfrac{n + \sin n^2}{n + \cos n}$; (iii) $\lim_{n\to\infty} \dfrac{1 - 2 + 3 - 4 + \cdots + (-2n)}{\sqrt{n^2 + 1}}$;

   (iv) $\lim_{n\to\infty} \prod_{k=2}^{n} \dfrac{k^3 - 1}{k^3 + 1}$; (v) $\lim_{n\to\infty} \dfrac{1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!}{(n+1)!}$.

5. Check whether the limit exists. If so find it.

   (i) $\lim_{x\to 0} x \cos \dfrac{1}{x}$; (ii) $\lim_{x\to 0} x \left[\dfrac{1}{x} \right]$; (iii) $\lim_{x\to 0} \dfrac{[x]}{x}$; (iv) $\lim_{x\to\infty} x\left(\sqrt{1 + x^2} - \sqrt[3]{x^3 + 1} \right)$;

   (v) $\lim_{x\to 0} \dfrac{\cos\left(\frac{\pi}{2} \cos x \right)}{\sin\left(\sin x \right)}$.

6. Assuming that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$, find

$$\lim_{x\to 0} \frac{\ln(1 + x)}{x}.$$

7. Find $\lim_{x\to 0} \dfrac{\ln \cos x}{\tan x^2}$.